

TORAL OR NON LOCALLY CONNECTED MINIMAL SETS FOR R -CLOSED SURFACE HOMEOMORPHISMS

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ABSTRACT. Let M be an orientable connected closed surface and f be an R -closed homeomorphism on M which is isotopic to identity. Then the suspension of f satisfies one of the following condition: 1) the closure of each element of it is toral. 2) there is a minimal set which is not locally connected. Moreover, we show that any positive iteration of an R -closed homeomorphism on a compact metrizable space is R -closed.

1. PRELIMINARIES

In this paper, we will show that if f is a nontrivial R -closed homeomorphism on M which is not periodic but isotopic to identity, then either a) f is “an irrational rotation” or b) there is a minimal set which is not locally connected. Moreover if M has genus ≥ 2 , then b) holds. Taking suspensions, we show that a) implies that each orbit closure is a torus. In addition, we show that any positive iteration of an R -closed homeomorphism on a compact metrizable space is R -closed.

For a subset U of a topological space, U is locally connected if every point of U admits a neighborhood basis consisting of open connected subsets. For a (binary) relation E on a set X (i.e. a subset of $X \times X$), let $E(x) := \{y \in X \mid (x, y) \in E\}$ for an element x of X . For a subset A of X , we say that A is E -saturated if $A = \cup_{x \in A} E(x)$. Also E define the relation \hat{E} on X with $\hat{E}(x) = \overline{E(x)}$. Recall that E is pointwise almost periodic if \hat{E} is an equivalence relation and E is R -closed if \hat{E} is closed. For an equivalence relation E , the collection of equivalence classes $\{E(x) \mid x \in X\}$ is a decomposition of X , denoted by \mathcal{F}_E . By a decomposition, we mean a family \mathcal{F} of pairwise disjoint subsets of a set X such that $X = \sqcup \mathcal{F}$. For a homeomorphism f on X , let E_f be the equivalent relation by $E_f(x) := \{f^k(x) \mid k \in \mathbb{Z}\}$. f is said to be R -closed if E_f is R -closed. Then f is R -closed if and only if $R := \{(x, y) \mid y \in \overline{O_f(x)}\}$ is closed. Note that f on a locally compact Hausdorff space is pointwise almost periodic if and only if \hat{E}_f is an equivalent relation (cf. Theorem 4.10 [GH]). We call that an equivalence relation E is L -stable if for an element x of X and for any open neighborhood U of $\hat{E}(x)$, there is a E -saturated open neighborhood V of $\hat{E}(x)$ contained in U . In [ES], they show the following: If a continuous mapping f of a topological space X in itself is either pointwise recurrent or pointwise almost periodic, then so is f^k for each positive integer k . In general cases, see Theorem 2.24, 4.04, and 7.04 [GH]. We show the following key lemma which is an R -closed version of this fact on a compact metrizable space.

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Lemma 1.1. *Let f an homeomorphism on a compact metrizable space X . If f is R -closed, then so is f^n for any $n \in \mathbb{Z}_{>0}$.*

Proof. Put $E := E_f$ and $E^n := E_{f^n}$. By Corollary 1.5 [?], we have that \hat{E} is an equivalence relation and so f are pointwise almost periodic. Since f is pointwise almost periodic, by Theorem 1[ES], we have that f^n is also pointwise almost periodic. Then \hat{E}^n is an equivalence relation. By Corollary 3.6[?], E is L -stable and it suffices to show that E^n is L -stable. Note that $E^n(x) \subseteq E(x)$ and so $\hat{E}^n(x) \subseteq \hat{E}(x)$. For $x \in X$ with $\hat{E}^n(x) = \hat{E}(x)$ and for any open neighborhood U of $\hat{E}(x) = \hat{E}^n(x)$, since E is L -stable, there is a E -saturated open neighborhood V of $\hat{E}(x)$ contained in U . Since $E^n(x) \subseteq E(x)$, we have that V is also a E^n -saturated open neighborhood V of $\hat{E}(x)$. Fix any $x \in X$ with $\hat{E}(x) \neq \hat{E}^n(x)$. Put $\{\hat{E}_1, \dots, \hat{E}_k\} := \{\hat{E}(f^k(x)) \mid k = 0, 1, \dots, n-1\}$ such that $\hat{E}_1 = \hat{E}^n(x)$ and $\hat{E}_i \cap \hat{E}_j = \emptyset$ for any $i \neq j \in \{1, \dots, k\}$. Let $\hat{E}' = \hat{E}_2 \sqcup \dots \sqcup \hat{E}_k$. Then \hat{E}_1 and \hat{E}' are closed and $\hat{E} = \hat{E}_1 \sqcup \dots \sqcup \hat{E}_k = \hat{E}_1 \sqcup \hat{E}'$. For any sufficiently small $\varepsilon > 0$, let $U_{1,\varepsilon} = B_\varepsilon(\hat{E}_1)$ (resp. $U'_\varepsilon = B_\varepsilon(\hat{E}')$) be the open ε -ball of \hat{E}_1 (resp. \hat{E}'). Since ε is small and X is normal, we obtain $U_{1,\varepsilon} \cap U'_\varepsilon = \emptyset$, $\overline{U_{1,\varepsilon/2}} \subseteq U_{1,\varepsilon}$, and $\overline{U'_{\varepsilon/2}} \subseteq U'_\varepsilon$. Since E is L -stable, there are neighborhoods $V_{1,\varepsilon} \subseteq U_{1,\varepsilon/2}$ (resp. $V'_\varepsilon \subseteq U'_{\varepsilon/2}$) of \hat{E}_1 (resp. \hat{E}') such that $V_{1,\varepsilon} \sqcup V'_\varepsilon$ is an E -saturated neighborhood of $\hat{E}(x)$. Since \hat{E}_1 and \hat{E}' are f^n -invariant and compact, there is a small $\delta > 0$ such that $f^n(V_{1,\delta}) \subseteq U_{1,\varepsilon}$ and $f^n(V'_\delta) \subseteq U'_\varepsilon$. Since $V_{1,\delta} \sqcup V'_\delta$ is f^n -invariant and $U_{1,\varepsilon} \cap U'_\varepsilon = \emptyset$, we obtain $V_{1,\delta} \sqcup V'_\delta = f^n(V_{1,\delta} \sqcup V'_\delta) = f^n(V_{1,\delta}) \sqcup f^n(V'_\delta)$, $f^n(V_{1,\delta}) \cap V'_\delta = \emptyset$, and $f^n(V'_\delta) \cap V_{1,\delta} = \emptyset$. Hence $V_{1,\delta} = f^n(V_{1,\delta})$ and $V'_\delta = f^n(V'_\delta)$. This implies that $V_{1,\delta}$ is an E^n -saturated neighborhood of $\hat{E}_1 = \hat{E}^n(x)$ with $V_{1,\delta} \subseteq U_{1,\varepsilon} = B_\varepsilon(\hat{E}_1) = B_\varepsilon(\hat{E}^n(x))$. \square

Note this lemma is not true for compact T_1 spaces. (e.g. a homeomorphism f on a non-Hausdorff 1-manifold $X = \{0_-, 0_+\} \sqcup]0, 1]$ by $f(0_\pm) = 0_\mp$ and $f|_{]0,1]} = \text{id}$).

2. MAIN RESULTS

From now on, let M be an orientable connected closed surface and f a nontrivial R -closed homeomorphism on M which is not periodic but isotopic to identity. We call that f on S^2 is a topological irrational rotation if there is an irrational number $\theta_0 \in \mathbb{R} - \mathbb{Q}$ such that f is topologically conjugate to a map on a unit sphere in \mathbb{R}^3 with the Cylindrical Polar Coordinates by $(\rho, \theta, z) \rightarrow (\rho, \theta + \theta_0, z) \in \mathbb{R}_{\geq 0} \times S^1 \times \mathbb{R}$. Also f on \mathbb{T}^2 is a topological irrational rotation if there is an irrational number $\theta_0 \in \mathbb{R} - \mathbb{Q}$ such that some positive iteration of f is topologically conjugate to a map $S^1 \times S^1 \rightarrow S^1 \times S^1$ by $(\theta, \varphi) \rightarrow (\theta + \theta_0, \varphi)$.

Lemma 2.1. *If every minimal set is locally connected, then f is a topological irrational rotation on $M = S^2$ or \mathbb{T}^2 .*

Proof. By Theorem 1 and Theorem 2 [BNW], since f is pointwise almost periodic, every orbit closure is a finite subset or a finite disjoint union of simple closed curves. We will show that there is a finite disjoint union of simple closed curves. Otherwise f is pointwise periodic. By [M], we have f is periodic, which contradicts. By Lemma 1.1, we have that f^n is also R -closed for any $n \in \mathbb{Z}_{>0}$. Hence there is an positive integer n such that f^n has a simple closed curve as a minimal set. Then Theorem 2.4 [Y2] implies that M is either \mathbb{T}^2 or S^2 . By Corollary 2.5 [Y2], if $M = S^2$, then

f has a null homotopic circle and so $n = 1$. Suppose $M = \mathbb{T}^2$ (resp. S^2). By Theorem 2.4[Y2], the set $\mathcal{F}_{\hat{E}_{f^n}}$ of orbits closures consists of essential circles (resp. two singular points and other circles). Fix any $x \in M$. Then $A := \mathbb{T}^2 - \hat{E}_{f^n}(x)$ (resp. $A := S^2 - \text{Sing}(f^n)$) is an open annulus. By Lemma 1.5 (resp. the proof of Lemma 2.1) [Y2], we have that the restriction $f^n|_A$ to the open annulus A is an irrational rotation. \square

This implies our main results.

Theorem 2.2. *Let M be an orientable connected closed surface and f be a nontrivial R -closed homeomorphism on M which is not periodic but isotopic to identity. Then one of the following holds:*

- 1) *f is a topological irrational rotation.*
 - 2) *there is a minimal set which is not locally connected.*
- Moreover 2) holds when M has genus ≥ 2 .

Taking a suspension, we have a following corollary.

Corollary 2.3. *Let M be an orientable connected closed surface and f be an R -closed homeomorphism on M which is isotopic to identity. Then the suspension of f satisfies one of the following condition:*

- 1) *the closure of each element of it is toral.*
- 2) *there is a minimal set which is not locally connected.*

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